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Central charge and BPS equations of $\mathcal{N} = 2$ supersymmetric Yang-Mills theory

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Abstract: We present a canonical derivation of the central charge and BPS equations of the low-energy effective action for $\mathcal{N} = 2$ SU(2) Yang-Mills theory in the Coulomb phase. Our formula for the central charge includes additional terms that depend on the core structure of charged solitons. We also show that supersymmetry is in general violated by magnetic charges. However, for BPS states, supersymmetry can be restored by adding source terms to the lagrangian.

Keywords: Extended Supersymmetry, Supersymmetric Effective Theories.

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Contents

1. Introduction

The remarkable solvability properties of $\mathcal{N} = 2$ gauge theories have led to much insight into the structure of four-dimensional field theories. In particular, Seiberg and Witten's exact solution of the low-energy effective lagrangian and BPS spectrum of the Coulomb phase of $\mathcal{N} = 2$ SU(2) Yang-Mills theory [1, 2], and subsequent work, has provided us with solvable examples of gauge theories exhibiting many of the strong–coupling phenomena associated with QCD.

In the Seiberg-Witten solution, a crucial role is played by the central charge Z of the $\mathcal{N} = 2$ algebra, which appears in the BPS inequality [3, 4]

$$M \geq \sqrt{2}|Z| \tag{1.1}$$

The masses of BPS states saturate this inequality, and are therefore determined by the central charge, which in turn depends on the vevs and mass parameters that parameterize the moduli space of Coulomb vacua. As given by Seiberg and Witten for pure SU(2) broken in the Coulomb phase to U(1), the formula for the central charge is

$$Z = a_\infty n_e + a_{D\infty} n_m \tag{1.2}$$

where n_e and n_m are the electric and magnetic charges, a_∞ is the asymptotic expectation value of the scalar component a of the unbroken U(1) vector multiplet far from all charges, and $a_{D\infty}$ is the asymptotic value of the dual scalar, a_D, which is expressed in terms of the $\mathcal{N}=2$ prepotential $\mathcal{F}(a)$ as $a_D \equiv \partial\mathcal{F}/\partial a$. Thus, the central charge determines $\mathcal{F}(a)$ and with it, the low-energy effective action to second order in derivatives.

By following the dependence of Z on a_∞, one finds that the BPS spectrum changes discontinuously along codimension 1 surfaces of marginal stability, and that on codimension 2 surfaces in the moduli space, certain states become massless. Each massless surface is fixed by the monodromy that results when one traverses a circuit enclosing the massless surface. Together, these monodromies generate the duality group, which combined with holomorphicity and weak-coupling data, completely determines the $\mathcal{N}=2$ prepotential.

Despite the importance of the central charge formula (1.2) in the Seiberg–Witten solution, a complete derivation has not previously been given. Seiberg and Witten stated it without proof, and checked only that magnetic monopoles in the SU(2) effective theory (extended to include W-bosons) saturate the inequality (1.1) with $Z = a_D n_m$, obtained *à la* BPS by bounding the bosonic part of the effective hamiltonian. Subsequent work by Chalmers, Roček, and von Unge [5] also verified eq. (1.1) for dyons, with Z as in eq. (1.2), and strongly suggested that the central charge should be given by eq. (1.2) in general. However, a more careful recent analysis by Argyres and Narayan [6] found additional terms in the central charge formula, which can only be set to zero by imposing special core boundary conditions on charged solitons.

Arguments based on the bosonic part of the effective hamiltonian are not a substitute for a direct and rigorous calculation of the central charge, which requires the construction of the superalgebra generators and the evaluation of their anticommutators. This approach also allows a a derivation of the BPS equations directly from supersymmetry, and ensures that solutions of the BPS equations are indeed supersymmetric.

A detailed computation of Z of this sort, following [4], was presented by Alvarez-Gaumé and Hassan [7], for $\mathcal{N}=2$ pure gauge theory with trivial prepotential. More recently, Iorio [8] has given a derivation of the central charge of the low-energy $\mathcal{N}=2$ SU(2) gauge theory, with arbitrary prepotential. This derivation is deficient in several respects: constrained variables are not treated canonically, the Bianchi identity is improperly applied in the presence of magnetic charges, and additional terms are thrown away without justification.

Wolf [10] has performed a computation of the central charge in the "high-energy" SU(2) theory, that is, with *massless* W-boson multiplets included in the action. His derivation correctly employs Dirac brackets, and reproduces the Seiberg-Witten formula (1.2). In this nonabelian context, some subtleties we shall encounter in the low-energy U(1) theory do not arise.

We begin in section 2 with a canonical derivation of the Noether supercharges. The application of Noether's theorem is straightforward and agrees with the result of [8]. One interesting result we will obtain along the way is that the supersymmetry variation of the action itself is not generally zero in the presence of magnetic charges. Specifically, the variation includes a term proportional to the magnetic charge density, which is not a total derivative.

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In section 3, we list the second-class constraints and derive Dirac brackets for the component fields and their conjugate momenta. We then use the results of sections 2 and 3 to compute, in section 4, the Dirac bracket of the two supercharges, and thus the central charge. In terms of the local electric and magnetic number densities ν_e and ν_m, we will find

$$Z = a_\infty\, n_e \,+\, a_{D\infty}\, n_m - \int d^3x\, (a\nu_e + a_D\nu_m)\,, \tag{1.3}$$

as claimed by Argyres and Narayan [6].

In section 5, we obtain BPS field equations by demanding invariance under an $\mathcal{N}=1$ supersymmetry subalgebra. We also determine which half of the $\mathcal{N}=2$ algebra is left unbroken by a given BPS state. Finally, in section 6 we discuss a method for adding source terms to the lagrangian. When these terms are included, the additional term in the supersymmetry variation that was found in section 2 is canceled, and the lagrangian becomes manifestly supersymmetric in the presence of magnetic charges.

2. U(1) effective lagrangian, equations of motion, and supersymmetry generators

We begin by writing down the U(1) effective lagrangian and equations of motion, from which we derive the canonical $\mathcal{N}=2$ supersymmetry generators using Noether's theorem. We shall find that the supersymmetry variation of the lagrangian contains an additional term which explicitly breaks some of the supersymmetry when magnetic charges are present.

The low-energy effective lagrangian of the $\mathcal{N}=2$ super Yang-Mills theory in the Coulomb phase, with gauge group SU(2) spontaneously broken down to U(1), has the following expression in terms of $\mathcal{N}=1$ superfields [1]:

$$\mathcal{L} = \frac{1}{4\pi}\,\mathrm{Im}\left[\int d^2\theta \int d^2\bar\theta \frac{\partial\mathcal{F}}{\partial\Phi}\bar\Phi + \frac{1}{2}\int d^2\theta \frac{\partial^2\mathcal{F}}{\partial\Phi^2} W^\alpha W_\alpha\right]. \tag{2.1}$$

Our notations and conventions are introduced in appendix A. As shown in appendix B, the lagrangian (2.1) has the following expression in terms of component fields

$$\begin{aligned}\mathcal{L} = \frac{1}{4\pi}\,\mathrm{Im}\Big\{\tau\Big[&\partial_\mu a\partial^\mu\bar a - i\psi\sigma^\mu\partial_\mu\bar\psi - i\lambda\sigma^\mu\partial_\mu\bar\lambda - \frac{1}{4}F_{\mu\nu}(F^{\mu\nu} - i\widetilde{F}^{\mu\nu}) + f\bar f + \frac{1}{2}D^2\Big] + \\ &+ \frac{\tau'}{2}\Big[i\sqrt{2}D(\psi\lambda) - \sqrt{2}(\lambda\,\sigma^{\mu\nu}\psi)F_{\mu\nu} - (\psi\psi)\bar f - (\lambda\lambda)f\Big] + \frac{\tau''}{4}(\psi\psi)(\lambda\lambda)\Big\}.\end{aligned} \tag{2.2}$$

The auxiliary fields f and D can be eliminated using their constraint equations:

$$f = \frac{1}{4i}\frac{1}{\mathrm{Im}\,\tau}\left(\tau'\psi\psi - \bar\tau'\bar\lambda\bar\lambda\right), \qquad D = -\frac{1}{2\sqrt{2}}\frac{1}{\mathrm{Im}\,\tau}\left(\tau'\psi\lambda + \bar\tau'\bar\psi\bar\lambda\right).$$

The equation of motion for the gauge field A_ν is

$$-\frac{1}{4\pi}\partial_\mu\,\mathrm{Im}\left[\tau(F^{\mu\nu} - i\widetilde{F}^{\mu\nu}) + \sqrt{2}\tau'(\lambda\sigma^{\mu\nu}\psi)\right] = 0\,, \qquad \nu = 0,1,2,3\,. \tag{2.3}$$

The equation of motion obtained by varying the scalar field $\bar{a}$ can be written

$$\partial_\mu \tau \partial^\mu a + 2i\,(\mathrm{Im}\tau)\,\partial_\mu \partial^\mu a - \frac{1}{4}\bar{\tau}' F_{\mu\nu}\left(F^{\mu\nu} + i\widetilde{F}^{\mu\nu}\right) + \cdots = 0 \tag{2.4}$$

where the dots refer to terms involving fermion fields.

Using the variations of the component fields under an $\mathcal{N}=1$ transformation with parameter ξ, we eventually obtain (see appendix C) the $\mathcal{N}=1$ supersymmetry variation of the lagrangian,

$$\begin{aligned}\delta\mathcal{L} = \partial_\mu \Bigg\{ \frac{1}{4\pi}\,\mathrm{Im}\Big[&\sqrt{2}\tau\,(\xi\psi)\,\partial^\mu\bar{a} - 2\sqrt{2}\bar{\tau}\,(\xi\sigma^{\mu\nu}\psi)\,\partial_\nu\bar{a} - i\sqrt{2}\bar{\tau}'\,(\bar{\psi}\bar{\lambda})(\xi\,\sigma^\mu\bar{\lambda}) + \\ &+ i\bar{\tau}\,(\xi\sigma_\nu\bar{\lambda})\left(F^{\mu\nu} + i\widetilde{F}^{\mu\nu}\right) - i\sqrt{2}\bar{\tau}f\,(\xi\sigma^\mu\bar{\psi}) + \bar{\tau}D\,(\xi\,\sigma^\mu\bar{\lambda})\Big]\Bigg\} + \\ + \frac{1}{4\pi}\,\mathrm{Im}\left[2\bar{\tau}\,(\xi\sigma^\nu\bar{\lambda})\left(\partial^\mu \widetilde{F}_{\mu\nu}\right)\right]. &\end{aligned} \tag{2.5}$$

The last term in (2.5) remains outside the total divergence. In the absence of magnetic charges, it vanishes by the Bianchi identity. However, in the present theory, magnetic monopoles do exist and we have, in terms of the magnetic number density ν_m,

$$\partial_i \widetilde{F}^{i0} = 4\pi\nu_m\,.$$

Thus the last term in (2.5) cannot be ignored and appears to break supersymmetry. However, as will be shown in section 6, a consistent framework can be formulated in which supersymmetry is manifest even in the presence of monopoles, and this term does not contribute.

By Noether's theorem [11], the first of the two supercurrents is given by

$$\begin{aligned} S^{(1)\mu} &= \left[\delta^{(1)}a\frac{\partial\mathcal{L}}{\partial(\partial_\mu a)} + \delta^{(1)}\psi\frac{\partial\mathcal{L}}{\partial(\partial_\mu\psi)} + \delta^{(1)}\lambda\,\frac{\partial\mathcal{L}}{\partial(\partial_\mu\lambda)}\right] + \text{c.c.} \\ &\quad + \delta^{(1)}A_\nu\,\frac{\partial\mathcal{L}}{\partial(\partial_\mu A_\nu)} - K^{(1)\mu} \\ &= \frac{1}{4\pi}\Bigg\{\sqrt{2}(\mathrm{Im}\,\tau)(\psi\sigma^\mu\bar{\sigma}^\nu\xi^{(1)})\partial_\nu\bar{a} + \frac{1}{\sqrt{2}}\bar{\tau}'(\bar{\psi}\bar{\lambda})(\xi^{(1)}\sigma^\mu\bar{\lambda}) - \\ &\qquad - i(\xi^{(1)}\sigma_\nu\bar{\lambda})\left[(\mathrm{Im}\ \tau)(F^{\mu\nu} - i\widetilde{F}^{\mu\nu}) + \mathrm{Im}(\sqrt{2}\tau'\lambda\sigma^{\mu\nu}\psi)\right]\Bigg\} + \text{c.c.} \end{aligned} \tag{2.6}$$

where $\delta\mathcal{L} \equiv \partial_\mu K^\mu$ and $\delta^{(1)}$ denotes the first supersymmetry variation, as given in appendix C. Notice that the auxiliary fields do not appear in the supercurrent.

We construct the second $\mathcal{N}=1$ supercurrent by using the $\mathrm{SU}(2)_R$ symmetry[1] of the $\mathcal{N}=2$ vector multiplet. In the expression (2.6) we make the transformations

$$\xi^{(1)} \to \xi^{(2)}\,, \qquad \lambda \to \psi\,, \qquad \psi \to -\lambda\,,$$

[1]This differs from the treatment in [8] which uses $\mathrm{U}(2)_R$ symmetry combined with charge conjugation to construct a second supercurrent, leading to different transformation properties for the auxiliary fields.

(a and A^μ are singlets) and obtain

$$S^{(2)\mu} = \frac{1}{4\pi}\Big\{-\sqrt{2}\,(\mathrm{Im}\,\tau)\,\left(\lambda\sigma^\mu\bar\sigma^\nu\xi^{(2)}\right)\partial_\nu\bar a - \frac{1}{\sqrt{2}}\bar\tau'\,(\bar\psi\bar\lambda)\left(\xi^{(2)}\sigma^\mu\bar\psi\right)$$
$$- i\left(\xi^{(2)}\sigma_\nu\bar\psi\right)\left[(\mathrm{Im}\,\tau)\left(F^{\mu\nu} - i\widetilde{F}^{\mu\nu}\right) + \mathrm{Im}\left(\sqrt{2}\tau'\lambda\sigma^{\mu\nu}\psi\right)\right]\Big\} + \text{c.c.}\,.$$

The supercurrents may be decomposed into chiral and antichiral components

$$S^{(a)\mu} = \xi^{(a)\alpha}S^{(a)\mu}_\alpha + \bar\xi^{(a)}_{\dot\alpha}\bar S^{(a)\mu\dot\alpha}\,, \qquad a = 1, 2\,.$$

In order to derive the superalgebra, we will need to express the supercurrents in terms of canonical variables. To this end, we extract the following canonical momenta from the lagrangian (2.1):

$$\begin{aligned}
\Pi_a &= \frac{1}{4\pi}(\mathrm{Im}\,\tau)\partial^0\bar a\,,\\
(\Pi_\psi)^\beta &= \frac{1}{8\pi}\bar\tau\bar\psi_{\dot\beta}(\bar\sigma^0)^{\dot\beta\beta}\,, \qquad (\Pi_{\bar\psi})_{\dot\beta} = -\frac{1}{8\pi}\tau\psi^\beta(\sigma^0)_{\beta\dot\beta}\\
(\Pi_\lambda)^\beta &= \frac{1}{8\pi}\bar\tau\bar\lambda_{\dot\beta}(\bar\sigma^0)^{\dot\beta\beta}\,, \qquad (\Pi_{\bar\lambda})_{\dot\beta} = -\frac{1}{8\pi}\tau\lambda^\beta(\sigma^0)_{\beta\dot\beta}\\
\Pi_{A_\mu} &= -\frac{1}{4\pi}\,\mathrm{Im}\left[\tau\left(F^{0\mu} - i\widetilde{F}^{0\mu}\right) + \sqrt{2}\,\tau'\,(\lambda\,\sigma^{0\mu}\psi)\right]
\end{aligned} \tag{2.7}$$

The two generators of $\mathcal{N} = 2$ supersymmetry transformations are then

$$Q^{(1)}_\alpha = \int d^3x S^{(1)0}_\alpha(\mathbf{x}, t) = \int d^3x\Big[\sqrt{2}\Pi_a\psi_\alpha + \frac{\sqrt{2}}{4\pi}(\mathrm{Im}\,\tau)(\sigma^i\bar\sigma^0)_\alpha^{\ \delta}\psi_\delta\partial_i\bar a - \tag{2.8}$$
$$- \sqrt{2}\frac{\bar\tau'}{\bar\tau}(\bar\psi\bar\lambda)(\Pi_\lambda)_\alpha + i\left(\Pi_{A_i} - \frac{\bar\tau}{4\pi}\widetilde{F}^{0i}\right)(\sigma_i)_{\alpha\dot\alpha}\bar\lambda^{\dot\alpha}\Big]$$

$$Q^{(2)}_\beta = \int d^3y S^{(2)0}_\beta(\mathbf{x}, t) = \int d^3y\Big[-\sqrt{2}\Pi_a\lambda_\beta - \frac{\sqrt{2}}{4\pi}(\mathrm{Im}\,\tau)(\sigma^j\bar\sigma^0)_\beta^{\ \gamma}\lambda_\gamma\partial_j\bar a + \tag{2.9}$$
$$+ \sqrt{2}\frac{\bar\tau'}{\bar\tau}(\bar\psi\bar\lambda)(\Pi_\psi)_\beta + i\left(\Pi_{A_i} - \frac{\bar\tau}{4\pi}\widetilde{F}^{0i}\right)(\sigma_j)_{\beta\dot\beta}\bar\psi^{\dot\beta}\Big]\,.$$

3. Constraints and Dirac brackets

All commutators and anticommutators that will appear in the text are Dirac brackets, denoted by [,] or { , }. Poisson brackets, appearing only in this section, are distinguished by a subscript "P". The second-class constraints referred to below are relations between canonical momenta and fields. When such constraints are present in a system, the canonical quantization proceeds with Dirac brackets (to be defined below), rather than with Poisson brackets. For more details, we refer the reader to [12].

The second-class constraints for the lagrangian (2.1) are given by the Euler–Lagrange equations that do not involve time derivatives. We write each constraint as an expression set equal to zero, and assign to each of these expressions a label θ_i. For example, variation of the lagrangian with respect to ψ gives the constraint

$$(\theta_1)_\alpha = (\Pi_\psi)_\alpha + \frac{\bar\tau}{8\pi}(\sigma^0)_{\alpha\dot\beta}\bar\psi^{\dot\beta} = 0\,.$$

The second–class constraints for the lagrangian (2.1) are as follows:

$$
\begin{aligned}
(\theta_1)_\alpha &= (\Pi_\psi)_\alpha + \frac{\bar\tau}{8\pi}(\sigma^0)_{\alpha\dot\beta}\bar\psi^{\dot\beta}\,, \qquad (\theta_2)_{\dot\alpha} = (\Pi_{\bar\psi})_{\dot\alpha} + \frac{\tau}{8\pi}\psi^\beta(\sigma^0)_{\beta\dot\alpha}\,,\\
(\theta_3)_\alpha &= (\Pi_\lambda)_\alpha + \frac{\bar\tau}{8\pi}(\sigma^0)_{\alpha\dot\beta}\bar\lambda^{\dot\beta}\,, \qquad (\theta_4)_\alpha = (\Pi_{\bar\lambda})_{\dot\alpha} + \frac{\tau}{8\pi}\lambda^\beta(\sigma^0)_{\beta\dot\alpha}\,,\\
\theta_5 &= D + \frac{1}{2\sqrt{2}}\,\frac{1}{\operatorname{Im}\tau}(\tau'\psi\lambda - \bar\tau'\bar\psi\bar\lambda)\,,\\
\theta_6 &= f + \frac{i}{4\operatorname{Im}\tau}(\tau'\psi\psi - \bar\tau'\bar\lambda\bar\lambda)\,, \qquad \theta_7 = \bar f - \frac{i}{4\operatorname{Im}\tau}(\bar\tau'\bar\psi\bar\psi - \tau'\lambda\lambda)\,,\\
\theta_8 &= \Pi_f\,, \qquad \theta_9 = \Pi_D\,, \qquad \theta_{10} = \Pi_{\bar f}\,.
\end{aligned}
$$

In the Dirac procedure, these constraints are used to construct a nonsingular matrix C, with elements

$$c_{ij} = [\theta_i, \theta_j]_P$$

which are linear in the fermionic fields. In order to compute its inverse C^{-1}, we first have to expand C as:

$$C(q^i, \psi^\alpha) = C_0(q^i) + C_\alpha(q^i)\psi^\alpha\,,$$

where q^i and ψ^α are bosonic and fermionic variables, respectively. The inverse matrix, with elements c^{ij} (so that $c_{ij}c^{jk} = \delta_i{}^k$), is then given by

$$C^{-1} \equiv B = B_0 + B_\alpha\psi^\alpha\,,$$

where

$$B_0 = C_0^{-1}\,, \qquad B_\alpha = -C_0^{-1}C_\alpha C_0^{-1}\,.$$

Then, the Dirac bracket of any two fields or momenta, ϕ and χ, is defined in terms of the inverse matrix and canonical brackets as follows:

$$[\phi, \chi] = [\phi, \chi]_P - [\phi, \theta_j]_P c^{ji}[\theta_i, \chi]_P\,. \tag{3.1}$$

The matrix C is block diagonal, and in some cases it is sufficient to restrict attention to a submatrix. For example, since the supercharges (2.8) and (2.9) contain no auxiliary fields, the calculation of the central charge (section 4) does not involve the constraints $\theta_5, \ldots, \theta_{10}$, which can thus be left out of the matrix C.

Below we list some of the Dirac brackets needed in the calculation of the central charge:

$$
\begin{aligned}
[\Pi_a, \Pi_{\bar a}] &= -\delta^3\frac{1}{16\pi}\frac{\tau'\bar\tau'}{\operatorname{Im}\tau}(\psi\sigma^o\bar\psi + \lambda\sigma^o\bar\lambda)\\
[\Pi_a, \psi_\gamma] &= \delta^3\frac{\tau'}{2\operatorname{Im}\tau}\psi_\gamma\,, \qquad [\Pi_a, \lambda_\gamma] = \delta^3\frac{\tau'}{2\operatorname{Im}\tau}\lambda_\gamma\\
\left[\Pi_a, \bar\psi_{\dot\alpha}\right] &= \left[\Pi_a, \bar\lambda_{\dot\alpha}\right] = 0\,, \qquad [\Pi_{\bar a}, \lambda_\alpha] = 0\,, \qquad [\Pi_{\bar a}, \psi_\alpha] = 0\\
\left\{\psi_\alpha, \bar\psi^{\dot\alpha}\right\} &= \left\{\psi_\alpha, \bar\psi^{\dot\alpha}\right\} = -\delta^3\frac{4\pi}{\operatorname{Im}\tau}(\sigma^0)_{\alpha\dot\alpha}\\
\left\{\psi_\alpha, \Pi_{\psi\,\beta}\right\} &= \left\{\lambda_\alpha, \Pi_{\lambda\beta}\right\} = \delta^3\frac{\bar\tau}{2\operatorname{Im}\tau}\varepsilon_{\alpha\beta}
\end{aligned}
\tag{3.2}
$$

where δ^3 is an abbreviation for $\delta^{(3)}(\mathbf{x} - \mathbf{y})$.

The Dirac constraint formalism makes it possible to interchange freely the fields and their conjugate momenta, if their constraint relations are properly taken into account, allowing us to perform calculations in terms of either a combination of fields and momenta or just fields alone. (Incidentally, we do not find it necessary to adopt, for raising and lowering the spinor indices of momenta conjugated to fermion fields, a rule opposite to the one for the fields themselves, as in [8].)

4. Calculation of the central charge in the U(1) effective theory

We now compute the central charge of the $\mathcal{N}=2$ superalgebra, by evaluating the Dirac bracket of the two generators of $\mathcal{N}=2$ supersymmetry transformations (2.8) and (2.9). Expanding out eqs. (2.8) and (2.9), we obtain

$$\begin{aligned}
\left\{Q^{(1)}_\alpha(\mathbf{x}),Q^{(2)}_\beta(\mathbf{y})\right\} = \int d^3x\int d^3y\Big[& -2\left\{\Pi_a\psi_\alpha,\Pi_a\lambda_\beta\right\}- && (4.1)\\
& -\frac{2}{4\pi}(\sigma^i\bar\sigma^0)_\beta^{\ \gamma}\partial_i\bar a\left\{\Pi_a\psi_\alpha,(\operatorname{Im}\tau)\lambda_\gamma\right\}+ && (4.2)\\
& +i\sqrt{2}\Pi_a\left(\Pi_{A_i}-\frac{\bar\tau}{4\pi}\widetilde F^{0i}\right)(\sigma_i)_{\beta\dot\beta}\left\{\psi_\alpha,\bar\psi^{\dot\beta}\right\}+ && (4.3)\\
& +2\Pi_a\frac{\bar\tau'}{\bar\tau}\left\{\psi_\alpha,(\bar\psi\bar\lambda)(\Pi_\psi)_\beta\right\}- && (4.4)\\
& -\frac{2}{4\pi}(\sigma^i\bar\sigma^0)_\alpha^{\ \delta}\partial_i\bar a\left\{(\operatorname{Im}\tau)\psi_\delta,\Pi_a\lambda_\beta\right\}+ && (4.5)\\
& +\frac{i\sqrt2}{4\pi}(\operatorname{Im}\tau)(\sigma^i\bar\sigma^0)_\alpha^{\ \delta}\partial_i\bar a\left(\Pi_{A_j}-\frac{\bar\tau}{4\pi}\widetilde F^{0j}\right)(\sigma_j)_{\beta\dot\beta}\left\{\psi_\delta,\bar\psi^{\dot\beta}\right\}+ && (4.6)\\
& +\frac{2}{4\pi}(\operatorname{Im}\tau)\frac{\bar\tau'}{\bar\tau}(\partial_i\bar a)(\sigma^i\bar\sigma^0)_\alpha^{\ \delta}\left\{\psi_\delta,(\bar\psi\bar\lambda)(\Pi_\psi)_\beta\right\}- && (4.7)\\
& -i\sqrt2\Pi_a\left(\Pi_{A_i}-\frac{\bar\tau}{4\pi}\widetilde F^{0i}\right)(\sigma_i)_{\alpha\dot\alpha}\left\{\bar\lambda^{\dot\alpha},\lambda_\beta\right\}- && (4.8)\\
& -\frac{i\sqrt2}{4\pi}(\operatorname{Im}\tau)(\partial_j\bar a)\left(\Pi_{A_i}-\frac{\bar\tau}{4\pi}\widetilde F^{0i}\right)(\sigma_i)_{\alpha\dot\alpha}(\sigma^j\bar\sigma^0)_\beta^{\ \gamma}\left\{\bar\lambda^{\dot\alpha},\lambda_\gamma\right\}+ && (4.9)\\
& +\frac{1}{4\pi}\bar\tau(\sigma_i)_{\alpha\dot\alpha}\bar\lambda^{\dot\alpha}(\sigma_j)_{\beta\dot\beta}\bar\psi^{\dot\beta}\left[\Pi_{A_i},\widetilde F^{0j}\right]+ && (4.10)\\
& +\frac{1}{4\pi}\bar\tau(\sigma_i)_{\alpha\dot\alpha}\bar\lambda^{\dot\alpha}(\sigma_j)_{\beta\dot\beta}\bar\psi^{\dot\beta}\left[\widetilde F^{0i},\Pi_{A_j}\right]+ && (4.11)\\
& +2\Pi_a\frac{\bar\tau'}{\bar\tau}\left\{(\bar\psi\bar\lambda)(\Pi_\lambda)_\alpha,\lambda_\beta\right\}+ && (4.12)\\
& +\frac{2}{4\pi}(\operatorname{Im}\tau)\frac{\bar\tau'}{\bar\tau}(\partial_j\bar a)(\sigma^j\bar\sigma^0)_\beta^{\ \gamma}\left\{(\bar\psi\bar\lambda)(\Pi_\lambda)_\alpha,\lambda_\gamma\right\}\Big]. && (4.13)
\end{aligned}$$

The terms (4.3), (4.6), (4.8), and (4.9) directly involve the Dirac brackets appearing in eqs. (3.2). The brackets appearing in (4.1), (4.2), (4.5), and (4.4) give, respectively,

$$\begin{aligned}
\left\{\Pi_a\psi_\alpha,\Pi_a\lambda_\beta\right\} &= -\delta^3(\mathbf{x}-\mathbf{y})\frac{\tau'\Pi_a}{2\operatorname{Im}\tau}\left\{\psi_\alpha,\lambda_\beta\right\}=0\\
\left\{\Pi_a\psi_\alpha,(\operatorname{Im}\tau)\lambda_\gamma\right\} &= -\delta^3(\mathbf{x}-\mathbf{y})\frac{\tau'}{2}\left\{\psi_\alpha,\lambda_\gamma\right\}=0
\end{aligned}$$

$$\begin{aligned}
\{(\operatorname{Im}\tau)\psi_\delta, \Pi_a\lambda_\beta\} &= -\delta^3(\mathbf{x}-\mathbf{y})\frac{\tau'}{2}\{\psi_\delta, \lambda_\beta\} = 0 \\
\{\psi_\alpha, (\bar{\psi}\bar{\lambda})(\Pi_\psi)_\beta\} &= -\delta^{(3)}(\mathbf{x}-\mathbf{y})\frac{4\pi}{\operatorname{Im}\tau}(\sigma^0)_{\alpha\dot{\beta}}\bar{\lambda}^{\dot{\beta}}(\Pi_\psi)_\beta + \delta^{(3)}(\mathbf{x}-\mathbf{y})(\bar{\psi}\bar{\lambda})\frac{\bar{\tau}}{2\operatorname{Im}\tau}\varepsilon_{\alpha\beta}\,.
\end{aligned}$$

In (4.10) and (4.11) it is useful to indicate the spatial coordinates explicitly:

$$\frac{1}{4\pi}\Big\{\bar{\tau}(\mathbf{y})\big(\sigma_i\bar{\lambda}(\mathbf{x})\big)_\alpha\big(\sigma_j\bar{\psi}(\mathbf{y})\big)_\beta\Big[\Pi_{A_i}(\mathbf{x}),\widetilde{F}^{0j}(\mathbf{y})\Big]+\bar{\tau}(\mathbf{x})\big(\sigma_i\bar{\lambda}(\mathbf{x})\big)_\alpha\big(\sigma_j\bar{\psi}(\mathbf{y})\big)_\beta\Big[\widetilde{F}^{0i}(\mathbf{x}),\Pi_{A_j}(\mathbf{y})\Big]\Big\}$$

For the lagrangian (2.1), the Dirac brackets for the gauge field are equal to the canonical brackets:

$$\begin{aligned}
\Big[\Pi_{A_i}(\mathbf{x}),\widetilde{F}^{0j}(\mathbf{y})\Big] &= \varepsilon^{0jkl}\partial_{y^k}\left[\Pi_{A_i}(\mathbf{x}),A_l(\mathbf{y})\right] = i\varepsilon^{0ijk}\partial_{y^k}\delta^3(\mathbf{x}-\mathbf{y}) \\
\Big[\widetilde{F}^{0i}(\mathbf{x}),\Pi_{A_j}(\mathbf{y})\Big] &= i\varepsilon^{0ijk}\partial_{x^k}\delta^3(\mathbf{x}-\mathbf{y})\,.
\end{aligned}$$

Integrating (4.10) by parts and dropping fermionic boundary terms (which vanish because the fermion fields vary with distance as $|\mathbf{x}|^{-3/2}$), we obtain

$$\begin{aligned}
\frac{i\varepsilon^{0ijk}}{4\pi}\int d^3x\,\big(\sigma_i\bar{\lambda}(\mathbf{x})\big)_\alpha\int d^3y\,\bar{\tau}(\mathbf{y})\,\big(\sigma_j\bar{\psi}(\mathbf{y})\big)_\beta\,\partial_{y^k}\delta^3(\mathbf{x}-\mathbf{y}) &= \\
= \frac{i\varepsilon^{0ijk}}{4\pi}\int d^3x\,\left[\partial_{x^k}\big(\sigma_i\bar{\lambda}(\mathbf{x})\big)_\alpha\right]\bar{\tau}(\mathbf{x})\,\big(\sigma_j\bar{\psi}(\mathbf{x})\big)_\beta\,.
\end{aligned}$$

Similarly, integrating (4.11) by parts,

$$\begin{aligned}
\frac{i\varepsilon^{0ijk}}{4\pi}\int d^3x\,\bar{\tau}(\mathbf{x})\,\big(\sigma_i\bar{\lambda}(\mathbf{x})\big)_\alpha\left[\partial_{x^k}\delta^3(\mathbf{x}-\mathbf{y})\right]\int d^3y\,\big(\sigma_j\bar{\psi}(\mathbf{y})\big)_\beta &= \\
= \frac{i\varepsilon^{0ijk}}{4\pi}\int d^3x\,\bar{\tau}(\mathbf{x})\,\big(\sigma_i\bar{\lambda}(\mathbf{x})\big)_\alpha\,\partial_{x^k}\big(\sigma_j\bar{\psi}(\mathbf{x})\big)_\beta\,.
\end{aligned}$$

From here on it is no longer necessary to indicate the dependence on $\mathbf{x}$. Adding the results from (4.10) and (4.11), we obtain

$$\frac{i\varepsilon^{0ijk}}{4\pi}\int d^3x\,\bar{\tau}\partial_k\Big(\big(\sigma_i\bar{\lambda}\big)_\alpha\big(\sigma_j\bar{\psi}\big)_\beta\Big) = -\frac{i\varepsilon^{0ijk}}{4\pi}\int d^3x\,(\partial_k\bar{\tau})\,\big(\sigma_i\bar{\lambda}\big)_\alpha\big(\sigma_j\bar{\psi}\big)_\beta$$

where in the last step we have again dropped a fermionic boundary term. Using the identities

$$(\sigma^\mu)_{\alpha\dot{\alpha}}(\sigma^\nu)_{\beta\dot{\beta}} - (\sigma^\nu)_{\alpha\dot{\alpha}}(\sigma^\mu)_{\beta\dot{\beta}} = 2\Big[(\sigma^{\mu\nu}\varepsilon)_{\alpha\beta}\,\varepsilon_{\dot{\alpha}\dot{\beta}} + (\varepsilon\bar{\sigma}^{\mu\nu})_{\dot{\alpha}\dot{\beta}}\,\varepsilon_{\alpha\beta}\Big] \tag{4.14}$$

and

$$\varepsilon^{\mu\nu\rho\lambda}\sigma_{\rho\lambda} = -2i\sigma^{\mu\nu}\,, \qquad \varepsilon^{\mu\nu\rho\lambda}\bar{\sigma}_{\rho\lambda} = 2i\bar{\sigma}^{\mu\nu}\,,$$

we can further write the sum of (4.10) and (4.11) as

$$\begin{aligned}
-\frac{i}{4\pi}\frac{1}{2}\varepsilon^{0ijk}\int d^3x\,(\partial_k\bar{\tau})\Big[\big(\sigma_i\bar{\lambda}\big)_\alpha\big(\sigma_j\bar{\psi}\big)_\beta - \big(\sigma_j\bar{\lambda}\big)_\alpha\big(\sigma_i\bar{\psi}\big)_\beta\Big] &= \\
= \frac{2}{4\pi}\int d^3x\,(\partial_k\bar{\tau})\Big[\big(\bar{\psi}\bar{\lambda}\big)\Big(\sigma^{0k}\Big)_\alpha^{\ \gamma}\varepsilon_{\gamma\beta} - \Big(\bar{\lambda}\bar{\sigma}^{0k}\bar{\psi}\Big)\varepsilon_{\alpha\beta}\Big]\,.
\end{aligned}$$

Finally, in (4.12) and (4.13) we have

$$\left\{\left(\bar{\psi}\bar{\lambda}\right)\left(\Pi_{\lambda}\right)_{\alpha},\lambda_{\beta}\right\} = -\delta^{(3)}\left(\mathbf{x}-\mathbf{y}\right)\frac{4\pi}{\operatorname{Im}\tau}\bar{\psi}_{\dot{\beta}}\left(\bar{\sigma}^{0}\right)^{\dot{\beta}\delta}\varepsilon_{\delta\beta}\left(\Pi_{\lambda}\right)_{\alpha} - \delta^{(3)}\left(\mathbf{x}-\mathbf{y}\right)\left(\bar{\psi}\bar{\lambda}\right)\frac{\bar{\tau}}{2\operatorname{Im}\tau}\varepsilon_{\alpha\beta}\,.$$

Collecting all these results, we find

$$\begin{aligned}
\left\{Q_{\alpha}^{(1)},Q_{\beta}^{(2)}\right\} = \int d^{3}x\Big[& i\sqrt{2}\left(\sigma^{i}\bar{\sigma}^{0}\right)_{\alpha}^{\;\delta}\left(\partial_{i}\bar{a}\right)\left(\Pi_{A_{j}}-\frac{\bar{\tau}}{4\pi}\widetilde{F}^{0j}\right)\left(\sigma_{j}\bar{\sigma}^{0}\right)_{\beta}^{\;\gamma}\varepsilon_{\gamma\delta} - && (4.15)\\
& -2\frac{\bar{\tau}'}{\bar{\tau}}\left(\partial_{i}\bar{a}\right)\left(\sigma^{i}\bar{\sigma}^{0}\right)_{\alpha}^{\;\delta}\left(\sigma^{0}\bar{\lambda}\right)_{\delta}\left(\Pi_{\psi}\right)_{\beta} + && (4.16)\\
& +\frac{\bar{\tau}'}{4\pi}\left(\partial_{i}\bar{a}\right)\left(\sigma^{i}\bar{\sigma}^{0}\right)_{\alpha}^{\;\delta}\left(\bar{\psi}\bar{\lambda}\right)\varepsilon_{\delta\beta} - && (4.17)\\
& -i\sqrt{2}\left(\sigma^{j}\bar{\sigma}^{0}\right)_{\beta}^{\;\delta}\left(\partial_{j}\bar{a}\right)\left(\Pi_{A_{i}}-\frac{\bar{\tau}}{4\pi}\widetilde{F}^{0i}\right)\left(\sigma_{i}\bar{\sigma}^{0}\right)_{\alpha}^{\;\gamma}\varepsilon_{\gamma\delta} + && (4.18)\\
& +\frac{2}{4\pi}\left(\partial_{k}\bar{\tau}\right)\left[\left(\bar{\psi}\bar{\lambda}\right)\left(\sigma^{0k}\right)_{\alpha}^{\;\gamma}\varepsilon_{\gamma\beta}-\left(\bar{\lambda}\bar{\sigma}^{0k}\bar{\psi}\right)\varepsilon_{\alpha\beta}\right] + && (4.19)\\
& +\frac{\bar{\tau}'}{4\pi}\left(\partial_{j}\bar{a}\right)\left(\sigma^{j}\bar{\sigma}^{0}\right)_{\beta}^{\;\delta}\left(\bar{\psi}\bar{\lambda}\right)\varepsilon_{\delta\alpha} - && (4.20)\\
& -2\frac{\bar{\tau}'}{\bar{\tau}}\left(\partial_{j}\bar{a}\right)\left(\sigma^{j}\bar{\sigma}^{0}\right)_{\beta}^{\;\gamma}\left(\bar{\psi}\bar{\sigma}^{0}\right)^{\delta}\varepsilon_{\delta\gamma}\left(\Pi_{\lambda}\right)_{\alpha}\Big]. && (4.21)
\end{aligned}$$

The terms (4.17) and (4.20) add up to

$$-\frac{4}{4\pi}\left(\partial_{i}\bar{\tau}\right)\left(\bar{\psi}\bar{\lambda}\right)\left(\sigma^{0i}\right)_{\alpha}^{\;\gamma}\varepsilon_{\gamma\beta} \qquad (4.22)$$

while (4.16) and (4.21) sum to

$$\frac{1}{4\pi}\left(\partial_{i}\bar{\tau}\right)\left[\left(\sigma^{i}\bar{\lambda}\right)_{\alpha}\left(\sigma^{0}\bar{\psi}\right)_{\beta}+\left(\sigma^{i}\bar{\psi}\right)_{\beta}\left(\sigma^{0}\bar{\lambda}\right)_{\alpha}\right]. \qquad (4.23)$$

Applying again the identity (4.14), we can combine (4.22) and (4.23) to obtain

$$\frac{2}{4\pi}\left(\partial_{i}\bar{\tau}\right)\left[\left(\bar{\lambda}\bar{\sigma}^{0i}\bar{\psi}\right)\varepsilon_{\alpha\beta}-\left(\bar{\psi}\bar{\lambda}\right)\left(\sigma^{0i}\right)_{\alpha}^{\;\gamma}\varepsilon_{\gamma\beta}\right],$$

which cancels (4.19). The only surviving terms are (4.15) and (4.18) which give

$$\left\{Q_{\alpha}^{(1)},Q_{\beta}^{(2)}\right\} = -i\left(2\sqrt{2}\right)\int d^{3}x\left(\partial_{i}\bar{a}\right)\left(\Pi_{A_{i}}-\frac{\bar{\tau}}{4\pi}\widetilde{F}^{0i}\right)\varepsilon_{\alpha\beta}\,. \qquad (4.24)$$

Let us now write the realization of the $\mathcal{N}=2$ algebra on massive states in the rest frame as [13]

$$\begin{aligned}
\left\{Q_{\alpha}^{(a)},\bar{Q}_{(b)\dot{\beta}}\right\} &= 2M\left(\sigma^{0}\right)_{\alpha\dot{\beta}}\delta_{b}^{a} && (4.25)\\
\left\{Q_{\alpha}^{(a)},Q_{\beta}^{(b)}\right\} &= -i\left(2\sqrt{2}\right)\varepsilon_{\alpha\beta}\varepsilon^{ab}\bar{Z} && (4.26)\\
\left\{\bar{Q}_{\dot{\alpha}}^{(a)},\bar{Q}_{\dot{\beta}}^{(b)}\right\} &= i(2\sqrt{2})\varepsilon_{\dot{\alpha}\dot{\beta}}\varepsilon^{ab}Z && (4.27)
\end{aligned}$$

where $a,b=1,2$.

Comparing (4.24) and (4.26) we obtain the following central charge formula

$$Z = \int d^3x\, (\partial_i a) \left(\Pi_{A_i} - \frac{\tau}{4\pi} \widetilde{F}^{0i} \right) . \tag{4.28}$$

Using $\tau = \partial a_D / \partial a$, we can rewrite this as

$$Z = \int d^3x \left[(\partial_i a)\, \Pi_{A_i} - \frac{1}{4\pi} (\partial_i a_D)\, \widetilde{F}^{0i} \right] .$$

Integrating by parts and identifying the electric and magnetic charge integers as in appendix A, we obtain

$$\begin{aligned} Z &= \int d^3x \left[\partial_i (a \Pi_{A_i}) - \frac{1}{4\pi} \partial_i \left(a_D \widetilde{F}^{0i} \right) \right] - \int d^3x \left[a \partial_i \Pi_{A_i} - \frac{1}{4\pi} a_D \partial_i \widetilde{F}^{0i} \right] \\ &= a_\infty n_e + a_{D\infty} n_m - & (4.29) \\ &\quad - \int d^3x\, (a \partial_i \Pi_{A_i}) + \frac{1}{4\pi} \int d^3x \left(a_D \partial_i \widetilde{F}^{0i} \right) & (4.30) \end{aligned}$$

where in the surface terms we have replaced the fields a and a_D by their constant, non-zero, asymptotic values, a_∞ and $a_{D\infty}$, respectively, on the sphere $r \to \infty$. The central charge thus differs from the conventional value by the two integral expressions in the last line. Finally, expressing the integrals (4.30) in terms of the electric and magnetic number densities $\nu_e(x)$ and $\nu_m(x)$, the central charge can be written as in eq. (1.3).

5. BPS equations

We now derive the BPS equations directly from the $\mathcal{N} = 2$ supersymmetry algebra, by requiring that the field variations vanish for some linear combination of the two supersymmetry generators.

Consider the following linear combinations of supersymmetry generators,

$$a_\alpha = \frac{1}{\sqrt{2}} \left[Q_\alpha^{(1)} + \eta \varepsilon_{\alpha\beta} \left(\bar{\sigma}^0 \right)^{\dot{\gamma}\beta} \bar{Q}_{\dot{\gamma}}^{(2)} \right] , \tag{5.1}$$

$$b_\alpha = \frac{1}{\sqrt{2}} \left[Q_\alpha^{(1)} - \eta \varepsilon_{\alpha\beta} \left(\bar{\sigma}^0 \right)^{\dot{\gamma}\beta} \bar{Q}_{\dot{\gamma}}^{(2)} \right] . \tag{5.2}$$

where η is a complex constant whose physical significance will be explained shortly. Using eqs. (4.25)–(4.27) we find

$$\left. \begin{array}{l} \{a_\alpha, \bar{a}_{\dot{\beta}}\} \\ \{b_\alpha, \bar{b}_{\dot{\beta}}\} \end{array} \right\} = (\sigma^0)_{\alpha\dot{\beta}} \left[M \left(1 + |\eta|^2 \right) \mp 2\sqrt{2}\, \mathrm{Im}(\eta Z) \right] . \tag{5.3}$$

Since $\{a_\alpha, \bar{a}_{\dot{\beta}}\}$ and $\{b_\alpha, \bar{b}_{\dot{\beta}}\}$ are semipositive definite operators, we obtain the mass bound

$$M \geq \frac{2\sqrt{2}\, |\mathrm{Im}(\eta Z)|}{1 + |\eta|^2} . \tag{5.4}$$

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States that saturate this mass bound are called BPS states. If equality holds in (5.4), then (taking the upper sign in (5.3), with no loss of generality)

$$\{a_\alpha, \bar{a}_{\dot\beta}\} = 0\,, \qquad \{a_\alpha, \phi\} = 0\,, \qquad \{\bar{a}_{\dot\alpha}, \phi\} = 0\,, \tag{5.5}$$

where $\phi = a, \lambda_\alpha, \psi_\alpha, A_\mu$, and the curly brackets denote commutators or anticommutators as appropriate. Thus, BPS states are left invariant under the action of half of the supersymmetry generators, either by (5.1) or by (5.2). For a BPS soliton, the variations of the field configurations will vanish for a particular linear combination of $\xi^{(1)}Q^{(1)}$ and $\xi^{(2)}Q^{(2)}$ and their conjugates, that is, with a particular linear relation between the supersymmetry parameters $\xi^{(1)}$ and $\xi^{(2)}$.

To find this linear relation, let us first express a general $\mathcal{N} = 2$ supersymmetry variation of a component field ϕ in terms of the generators a and b:

$$\begin{aligned}\delta\phi &= i\xi^{(1)\alpha}\left\{Q^{(1)}_\alpha, \phi\right\} + i\bar\xi^{(1)}_{\dot\alpha}\left\{\bar{Q}^{(1)\dot\alpha}, \phi\right\} + i\xi^{(2)\beta}\left\{Q^{(2)}_\beta, \phi\right\} + i\bar\xi^{(2)}_{\dot\beta}\left\{\bar{Q}^{(2)\dot\beta}, \phi\right\} \\ &= \frac{i}{\sqrt{2}}\Bigg[\left(\xi^{(1)\alpha} - \eta^{-1}\bar\xi^{(2)}_{\dot\beta}\left(\bar\sigma^0\right)^{\dot\beta\alpha}\right)\{a_\alpha, \phi\} + \left(-\bar\xi^{(1)\dot\alpha} + \bar\eta^{-1}\left(\bar\sigma^0\right)^{\dot\alpha\alpha}\xi^{(2)}_\alpha\right)\{\bar{a}_{\dot\alpha}, \phi\} + \\ &\qquad + \left(\xi^{(1)\alpha} + \eta^{-1}\bar\xi^{(2)}_{\dot\beta}(\bar\sigma^0)^{\dot\beta\alpha}\right)\{b_\alpha, \phi\} + \left(-\bar\xi^{(1)\dot\alpha} - \bar\eta^{-1}\left(\bar\sigma^0\right)^{\dot\alpha\alpha}\xi^{(2)}_\alpha\right)\{\bar{b}_{\dot\alpha}, \phi\}\Bigg]\,.\end{aligned} \tag{5.6}$$

Now specializing to a BPS state satisfying (5.5), we obtain

$$\delta\phi = \frac{i}{\sqrt{2}}\left[\left(\xi^{(1)\alpha} + \eta^{-1}\bar\xi^{(2)}_{\dot\beta}\left(\bar\sigma^0\right)^{\dot\beta\alpha}\right)\{b_\alpha, \phi\} + \left(-\bar\xi^{(1)\dot\alpha} - \bar\eta^{-1}\left(\bar\sigma^0\right)^{\dot\alpha\alpha}\xi^{(2)}_\alpha\right)\{\bar{b}_{\dot\alpha}, \phi\}\right].$$

This identifies the particular supersymmetry transformation which leaves the fields of a BPS solution invariant, $\delta\phi = 0$; it is the transformation for which the parameters satisfy the relation

$$\xi^{(1)\alpha} = -\eta^{-1}\bar\xi^{(2)}_{\dot\beta}\left(\bar\sigma^0\right)^{\dot\beta\alpha} \tag{5.7}$$

We now obtain the BPS equations of the U(1) effective theory by imposing eq. (5.7) on the $\mathcal{N} = 2$ susy variations of the component fields (see appendix C) and requiring that these vanish.

The first BPS constraint is derived from

$$\delta a = \sqrt{2}\left(\xi^{(1)}\psi - \xi^{(2)}\lambda\right) = 0 \tag{5.8}$$

and (5.7), giving $\xi^{(2)\alpha}\lambda_\alpha = -\eta^{-1}\bar\xi^{(2)}_{\dot\beta}(\bar\sigma^0)^{\dot\beta\alpha}\psi_\alpha$, for any $\xi^{(2)\alpha}$. In particular, this should also hold when $\xi^{(2)\alpha}$ is replaced by $i\xi^{(2)\alpha}$. We are then led to conclude that $\xi^{(2)\alpha}\lambda_\alpha = 0$, for any $\xi^{(2)\alpha}$, which implies that

$$\lambda_\alpha = 0\,, \qquad \psi_\alpha = 0\,. \tag{5.9}$$

(Eqs. (5.9) also follow from $\delta A_\mu = 0$.)

Next we impose (5.7) on $\delta\lambda_\alpha = 0$ to obtain[2]

$$F^{0i} - i\widetilde{F}^{0i} = i\sqrt{2}\eta\partial^i a\,, \tag{5.10}$$

$$\partial_0 a = 0\,, \tag{5.11}$$

[2]The auxiliary fields f and D, although present in the general $\mathcal{N} = 2$ variations of λ_α and ψ_α (appendix C), do not appear here because their constraint equations (section 2) involve fermion fields, which are equal to zero for BPS states, according to (5.9).

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while the $\mathcal{N}=2$ variation of ψ_α gives

$$F^{0i} - i\widetilde{F}^{0i} = i\sqrt{2}\bar{\eta}^{-1}\partial^i a\,, \tag{5.12}$$

$$\partial_0 a = 0\,. \tag{5.13}$$

For these two sets of equations to agree, η must be a phase factor.

In terms of the electric and magnetic fields (defined in appendix A), the BPS equation (5.10) can be written

$$E^i - iB^i = -\frac{i\sqrt{2}\eta}{g}\partial^i a\,. \tag{5.14}$$

Using the definition (2.7) of the canonical momentum Π_{A_i}, we can further rewrite this as

$$\Pi_{A_i} - \frac{\bar{\tau}}{4\pi}\widetilde{F}^{0i} = -\frac{i\sqrt{2}\eta}{4\pi}(\mathrm{Im}\,\tau)\partial^i a\,. \tag{5.15}$$

Comparison with (4.28) gives a useful expression for the central charge Z, in terms only of the field a,

$$\begin{aligned} Z &= \int d^3x\,(\partial_i a)\left(\Pi_{A_i} - \frac{\tau}{4\pi}\widetilde{F}^{0i}\right) \\ &= \frac{i\sqrt{2}\bar{\eta}}{4\pi}\int d^3x(\mathrm{Im}\,\tau)\partial_i a\partial^i\bar{a}\,, \end{aligned} \tag{5.16}$$

which identifies the phase η as

$$\eta = -i\frac{\bar{Z}}{|Z|}\,. \tag{5.17}$$

Then, eq. (5.4) assumes the familiar form $M \geq \sqrt{2}|Z|$.

We can obtain an equation involving only a by taking the imaginary part of the divergence of the BPS equations in the form of eq. (5.10):

$$\nabla^2\,\mathrm{Re}(\eta a) = -\frac{4\pi}{\sqrt{2}}\nu_m\,. \tag{5.18}$$

Multiplying eq. (5.15) through by τ and taking the imaginary part of its divergence gives

$$\nabla^2\,\mathrm{Re}(\eta a_D) = \frac{4\pi}{\sqrt{2}}\nu_e\,. \tag{5.19}$$

Notice that the signs in eqs. (5.18) and (5.19) are consistent with S-duality, i.e., $a \to a_D$, $a_D \to -a$ and $(n_m, n_e) \to (-n_e, n_m)$. It is easy to check that any solution of the BPS equations also satisfies the full equations of motion.

6. Sources and the resolution of a paradox

As it stands, the lagrangian (2.1) cannot reproduce eqs. (5.18)–(5.19), with sources on the right-hand side, because it only describes the free-field vector multiplet. However, the

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equations of motion in the presence of electric and magnetic sources can be derived by adding the following classical source term to the original lagrangian:

$$\begin{aligned}\mathcal{L}_{\text{source}} &= \text{Im}\left[\sqrt{2}\eta\left(a\nu_e + a_D\nu_m\right) - iA_\mu j_e^\mu\right] \\ &= \text{Im}\left[\sqrt{2}\eta\left(a\nu_e + a_D\nu_m\right) - iA_0\nu_e\right],\end{aligned} \tag{6.1}$$

where the second line is valid for static fields and sources. The equation of motion for the scalar field $\bar{a}$ then becomes

$$\partial_\mu\tau\partial^\mu a + 2i(\text{Im}\,\tau)\partial_\mu\partial^\mu a - \frac{1}{4}\bar{\tau}' F_{\mu\nu}\left(F^{\mu\nu} + i\widetilde{F}^{\mu\nu}\right) = -\bar{\eta}\sqrt{2}(4\pi)(\nu_e + \bar{\tau}\nu_m) \tag{6.2}$$

where the fermionic terms have been omitted. It is easy to show that eqs. (5.18)–(5.19) are consistent with (6.2).

The addition of this source term to the lagrangian solves the mystery of the susy–violating term which appeared in the $\mathcal{N}=1$ variation (2.5) of the lagrangian (2.1),

$$\frac{1}{4\pi}\,\text{Im}\left[2\bar{\tau}\left(\xi\sigma^\nu\bar{\lambda}\right)\left(\partial^\mu\widetilde{F}_{\mu\nu}\right)\right]. \tag{6.3}$$

If no magnetic charges are present, then the Bianchi identity can be applied and the lagrangian is invariant under supersymmetry.[3]. In contrast, in the presence of magnetic charges ($\nu_m \neq 0$), we cannot invoke the Bianchi identity because $\partial^\mu\widetilde{F}_{\mu 0} = 4\pi\nu_m$. [4] However, by adding $\mathcal{L}_{\text{source}}$ to the lagrangian (2.1), we obtain a consistent picture in which the $\mathcal{N}=2$ supersymmetry variation of $\mathcal{L}_{\text{source}}$ cancels exactly the $\mathcal{N}=2$ version of (6.3), without it being necessary to impose the Bianchi identity. The following proof of invariance uses the relation (5.7) between the two supersymmetry parameters $\xi^{(1)\alpha}$ and $\bar{\xi}^{(2)}_{\dot{\beta}}$, given in section 4, which is the key to the physical meaning of this cancellation. The situation is analogous to the fact that gauge invariance in electromagnetism is broken in the presence of a prescribed external source, but is restored if the source is included in the full lagrangian of the system.

In terms of the magnetic number density, (6.3) becomes, for a static distribution of magnetic charge,

$$2\nu_m\,\text{Im}\left[\bar{\tau}\left(\xi\sigma^0\bar{\lambda}\right)\right].$$

Under an $\mathcal{N}=2$ transformation satisfying eq. (5.7), i.e., which preserves a BPS state, the corresponding extra term in the variation of the lagrangian is

$$-2\nu_m\,\text{Im}\left[\tau\left(-\eta\xi^{(2)}\lambda + \eta\xi^{(1)}\psi\right)\right] \tag{6.4}$$

with $\xi^{(1)}$ and $\xi^{(2)}$ related by eq. (5.7). Consider now the variation of the source term $\mathcal{L}_{\text{source}}$ under this $\mathcal{N}=2$ transformation, which we write as

$$\delta\mathcal{L}_{\text{source}} = \delta\mathcal{L}_m + \delta\mathcal{L}_e$$

[3]Even in the simple case of the lagrangian for the $\mathcal{N}=1$ vector superfield [14], in order to obtain the variation of the lagrangian as a total divergence, the Bianchi identity must be imposed [15]. In fact, a term of the form (6.3) will always arise from lagrangians containing an $\mathcal{N}=1$ vector superfield.

[4]In this regard, the present calculation in the U(1) effective theory differs crucially from the calculation of the central charge in the non-abelian case[7, 15, 10], where the the Bianchi identity is always satisfied.

where

$$\delta\mathcal{L}_m = \sqrt{2}\nu_m \operatorname{Im}(\eta\tau\delta a) \tag{6.5}$$

and

$$\delta\mathcal{L}_e = -\nu_e\left[\delta A^0 + \frac{i}{\sqrt{2}}\left(\eta\delta a - \bar{\eta}\delta\bar{a}\right)\right]. \tag{6.6}$$

The reason for this splitting of $\delta\mathcal{L}_{\text{source}}$ is that the piece which cancels (6.4) is (6.5), while (6.6) vanishes. Indeed,

$$\delta\mathcal{L}_m = 2\nu_m \operatorname{Im}\left[\eta\tau\left(\xi^{(1)}\psi - \xi^{(2)}\lambda\right)\right]$$

cancels (6.4), while

$$\delta\mathcal{L}_e = -i\nu_e\left(\bar{\xi}^{(1)}\bar{\sigma}^0\lambda - \bar{\lambda}\bar{\sigma}^0\xi^{(1)} + \bar{\xi}^{(2)}\bar{\sigma}^0\psi - \bar{\psi}\bar{\sigma}^0\xi^{(2)} + \eta\xi^{(1)}\psi - \eta\xi^{(2)}\lambda - \bar{\eta}\bar{\xi}^{(1)}\bar{\psi} + \bar{\eta}\bar{\xi}^{(2)}\bar{\lambda}\right)$$

vanishes by using (5.7) in the first four terms of the right-hand side, which completes the proof.

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A. Notations and conventions

Metric, Pauli matrices, and spinors. We use a Minkowski metric with signature $(+---)$. The conventions for supersymmetry are those of [14], with a few exceptions that follow from our choice of metric and will be noted below. Our conventions are then the same as in [16], excepting the fact that we take $\varepsilon^{0123} = +1$, as in [14]. To be specific, with our choice of metric, the right-hand sides of several relations from [14], appendix B, will have an opposite sign.[5] Another difference is that with the $(+---)$ metric, we must define $\sigma^0 = \bar{\sigma}^0 = 1_{2\times 2}$ (which will be our choice), while $(-+++)$ require $\sigma^0 = \bar{\sigma}^0 = -1_{2\times 2}$ (as in [14]), and we cannot mix the signs between these two conventions because otherwise a relation that has a form independent of the signature of the metric, such as (B.8) from [14], would not be consistent.

[5]The relations from [14], appendix B, for which the right-hand side changes sign are (B.4), (B.5), (B.11), (B.14), (B.19). The sign also changes for the first term on the right-hand side of the second relation (B.9).

Superfields. The $\mathcal{N}=1$ chiral superfield in the lagrangian (2.1) has the following field content,

$$\Phi = a + \sqrt{2}\theta\psi + i\left(\theta\sigma^{\mu}\bar{\theta}\right)\partial_{\mu}a - \frac{i}{\sqrt{2}}(\theta\theta)\left(\partial_{\mu}\psi\sigma^{\mu}\bar{\theta}\right) - \\ -\frac{1}{4}(\theta\theta)\left(\bar{\theta}\bar{\theta}\right)\partial_{\mu}\partial^{\mu}a + (\theta\theta)f$$

(we denote the scalar component of this superfield by a, as in [1]). The corresponding antichiral superfield and dual superfield will be denoted by $\bar{\Phi}$ and Φ_D, respectively. The vector superfield in the Wess-Zumino gauge is

$$V_{WZ} = \theta\sigma^{\mu}\bar{\theta}A_{\mu} + i(\theta\theta)\,\bar{\theta}\bar{\lambda} - i\left(\bar{\theta}\bar{\theta}\right)\theta\lambda + \frac{1}{2}(\theta\theta)\left(\bar{\theta}\bar{\theta}\right)D\,.$$

Notice that this expresion is the one from [16], and differs from the one in [14, eq. (6.6)], by the sign of the first term. The reason for this sign difference is that, in going from (4.9) to (6.2) in [14], the sign of the term written there as $\theta\sigma^m\bar{\theta}v_m$ is conventionally changed from + to − . We choose not to make this change, hence we follow [16]. The consequences of this difference are:

1) in the field-strength superfield W_{α}, the term $i(\sigma^{\mu\nu}\theta)_{\alpha}F_{\mu\nu}$ is now preceded by a plus sign, while in [14] it has a minus sign,

2) in the supersymmetry variation $\delta\lambda$, the term in $F_{\mu\nu}$ is now preceded by a minus sign, instead of a plus sign as in [14], and

3) in the lagrangian (2.1), the term in $(\lambda\sigma^{\mu\nu}\psi)F_{\mu\nu}$ has a minus sign in our convention, instead of a plus sign.

Obviously, with either convention, the physical results are the same.

Electric and magnetic fields. The electric and magnetic fields and charge integers are defined by

$$E^i = \frac{F^{i0}}{g}\,, \qquad B^i = \frac{\widetilde{F}^{i0}}{g}\,,$$

and

$$n_e = \int d^3x\partial_i\Pi_{A_i}\,, \qquad n_m = \frac{1}{4\pi}\int d^3x\partial_i\widetilde{F}^{i0}\,,$$

respectively, and we write $\tau = \theta/2\pi + i(4\pi/g^2)$. As a consistency check of these expressions, we now show that the correct expression for the electric charge of a dyon in a θ vacuum [17] follows directly from the definitions of the charge integers and the canonical momentum (2.7). We have

$$n_e = \int d^3x\partial_i\Pi_{A_i} = -\frac{1}{4\pi}\int d^3x\partial_i\,\mathrm{Im}\left[\tau\left(F^{0i} - i\widetilde{F}^{0i}\right) + \sqrt{2}\tau'\left(\lambda\sigma^{0i}\psi\right)\right].$$

The fermionic boundary term vanishes, because the fermion fields vary with distance as $|\mathbf{x}|^{-3/2}$, and we are left with

$$n_e = -\frac{1}{4\pi}\int d^3x\partial_i\left[(\mathrm{Im}\,\tau)F^{0i}\right] + \frac{1}{4\pi}\int d^3x\partial_i\left[(\mathrm{Re}\,\tau)\widetilde{F}^{0i}\right].$$

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On the sphere at $r \to \infty$, τ has the constant value $\tau_\infty = \tau(a_\infty)$, so we have

$$n_e = \frac{1}{g_\infty} \int d^3x \partial_i \left(-\frac{F^{0i}}{g} \right) - (\mathrm{Re}\, \tau_\infty) n_m = \frac{1}{g_\infty} \int d^3x (\partial_i E^i) - \frac{\theta_\infty}{2\pi} n_m \,.$$

With the usual identification of the total electric charge $Q_e = \int d^3x \partial_i E^i$, we obtain the familiar expression for the electric charge of a dyon

$$Q_e = \left(n_e + \frac{\theta_\infty}{2\pi} n_m \right) g_\infty \,.$$

B. Derivation of the U(1) effective lagrangian

The lagrangian (2.1) is written in terms of $\mathcal{N} = 1$ superfields. This expression is derived from the $\mathcal{N} = 2$ chiral effective action [18]. The first term in (2.1) is the coefficient of the $(\theta\theta)(\bar{\theta}\bar{\theta})$ term (known as the D-term) in the expansion of the Kähler potential,

$$\int d^2\theta \int d^2\bar{\theta} K(\Phi, \bar{\Phi}) \,,$$

where Φ is the chiral superfield given above. Denoting $\Phi \equiv a + \Delta\Phi$ and expanding the Kähler potential in a Taylor series, we have

$$\begin{aligned} \int d^2\theta \int d^2\bar{\theta} K(\Phi, \bar{\Phi}) = & -\Bigg(\frac{1}{4} \frac{\partial K}{\partial a} \partial_\mu \partial^\mu a + \frac{1}{4} \frac{\partial^2 K}{\partial a^2} \partial_\mu a \partial^\mu a \\ & + \frac{i}{2} \frac{\partial^2 K}{\partial a \partial \bar{a}} \psi \sigma^\mu \partial_\mu \bar{\psi} + \frac{i}{2} \frac{\partial^3 K}{\partial a \partial \bar{a}^2} (\partial_\mu \bar{a})(\psi \sigma^\mu \bar{\psi}) + \frac{1}{2} \frac{\partial^3 K}{\partial a^2 \partial \bar{a}} (\psi\psi) \bar{f} \Bigg) + \text{c.c.} \\ & + \frac{\partial^2 K}{\partial a \partial \bar{a}} \left(\frac{1}{2} \partial_\mu a \partial^\mu \bar{a} + f \bar{f} \right) + \frac{1}{4} \frac{\partial^4 K}{\partial a^2 \partial \bar{a}^2} (\psi\psi)(\bar{\psi}\bar{\psi}) \,. \end{aligned} \tag{B.1}$$

The terms of (B.1) involving only a and $\bar{a}$ can be combined into

$$\frac{\partial^2 K}{\partial a \partial \bar{a}} \partial_\mu a \partial^\mu \bar{a} - \partial_\mu \left(\frac{1}{4} \frac{\partial K}{\partial a} \partial^\mu a + \text{c.c.} \right)$$

and we drop the total divergence.

In terms of the holomorphic prepotential $\mathcal{F}(\Phi)$ which depends only on the superfield Φ, the Kähler potential of the present theory can be expressed as

$$K(\Phi, \bar{\Phi}) = \mathrm{Im} \left(\frac{\partial \mathcal{F}}{\partial \Phi} \bar{\Phi} \right) .$$

For the Kähler metric we have

$$g_{a\bar{a}} = \mathrm{Im}\, \tau(a) = \mathrm{Im} \frac{\partial^2 \mathcal{F}}{\partial a^2} = \frac{\partial^2 K}{\partial a \partial \bar{a}} \,.$$

Also,

$$\frac{\partial^3 K}{\partial a^2 \partial \bar{a}} = \frac{1}{2i} \frac{\partial \tau}{\partial a} \,, \qquad \frac{\partial^3 K}{\partial a \partial \bar{a}^2} = -\frac{1}{2i} \frac{\partial \bar{\tau}}{\partial \bar{a}} \,, \qquad \frac{\partial^4 K}{\partial a^2 \partial \bar{a}^2} = 0 \,.$$

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The terms

$$-\left(\frac{i}{2}\frac{\partial^2 K}{\partial a\partial\bar{a}}\psi\sigma^\mu\partial_\mu\bar{\psi}+\frac{i}{2}\frac{\partial^3 K}{\partial a\partial\bar{a}^2}\left(\partial_\mu\bar{a}\right)\left(\psi\sigma^\mu\bar{\psi}\right)\right)+\text{c.c.}\,.$$

of (B.1) reduce to $\mathrm{Im}(-i\tau\psi\sigma^\mu\partial_\mu\bar{\psi})$, where we have dropped a total divergence. Then (B.1) becomes

$$\begin{aligned}\int d^2\theta\int d^2\bar{\theta}K(\Phi,\bar{\Phi}) &= \mathrm{Im}\int d^2\theta\int d^2\bar{\theta}\frac{\partial\mathcal{F}}{\partial\Phi}\bar{\Phi}=\\ &= \mathrm{Im}\left[\tau(a)\left(\partial_\mu a\partial^\mu\bar{a}-i\psi\sigma^\mu\partial_\mu\bar{\psi}+f\bar{f}\right)-\frac{1}{2}\tau'(\psi\psi)\bar{f}\right], \quad \text{(B.2)}\end{aligned}$$

where $\tau'\equiv\partial\tau/\partial a$.

The second term in the lagrangian (2.1) is the coefficient of the $(\theta\theta)$ term (known as the F-term) in the expansion of $\tau(\Phi)W^\alpha W_\alpha$, where

$$\tau(\Phi)=\frac{\partial^2\mathcal{F}}{\partial\Phi^2}=\tau(a)+\frac{\partial\tau}{\partial a}\Delta\Phi+\frac{1}{2}\frac{\partial^2\tau}{\partial a^2}(\Delta\Phi)^2$$

(all higher-order terms vanish). We have

$$\begin{aligned}\int d^2\theta\tau(\Phi)W^\alpha W_\alpha &= \tau\left[D^2-2i\lambda\sigma^\mu\partial_\mu\bar{\lambda}-\frac{1}{2}F_{\mu\nu}(F^{\mu\nu}-i\widetilde{F}^{\mu\nu})\right]+\\ &\quad+\tau'\left[i\sqrt{2}D(\psi\lambda)-\sqrt{2}(\lambda\sigma^{\mu\nu}\psi)F_{\mu\nu}-(\lambda\lambda)f\right]+\\ &\quad+\frac{1}{2}\tau''(\lambda\lambda)(\psi\psi)\,. \quad \text{(B.3)}\end{aligned}$$

(Note that in this formulation the component fields depend on the *superspace* coordinate $y^\mu=x^\mu+i\theta\sigma^\mu\bar{\theta}$.)

Using (B.2) and (B.3) in (2.1), we obtain the effective lagrangian in terms of component fields.

C. $\mathcal{N}=1$ supersymmetry variation of the $\mathrm{U}(1)$ effective lagrangian

The variations of the component fields of the lagrangian (2.1) under an $\mathcal{N}=1$ transformation with parameter ξ are

$$\begin{aligned}\delta a &= \sqrt{2}\xi\psi \qquad \left(\text{so } \delta\tau=\sqrt{2}\tau'\xi\psi\right), \qquad \delta\bar{a} = \sqrt{2}\bar{\xi}\bar{\psi}\,,\\ \delta\psi &= i\sqrt{2}\sigma^\mu\bar{\xi}\partial_\mu a+\sqrt{2}\xi f\,,\\ \delta\bar{\psi} &= i\sqrt{2}\bar{\sigma}^\mu\xi\partial_\mu\bar{a}+\sqrt{2}\bar{\xi}\bar{f}\,,\\ \delta f &= i\sqrt{2}\bar{\xi}\bar{\sigma}^\mu\partial_\mu\psi\,, \qquad \delta\bar{f} = i\sqrt{2}\xi\sigma^\mu\partial_\mu\bar{\psi}\,,\\ \delta A^\mu &= i\left(\bar{\xi}\bar{\sigma}^\mu\lambda-\bar{\lambda}\bar{\sigma}^\mu\xi\right), \qquad \text{or}\\ \delta F^{\mu\nu} &= i\left[\left(\xi\sigma^\nu\partial^\mu\bar{\lambda}+\bar{\xi}\bar{\sigma}^\nu\partial^\mu\lambda\right)-(\nu\longleftrightarrow\mu)\right],\\ \delta\lambda &= -\frac{1}{2}\sigma^\mu\bar{\sigma}^\nu\xi F_{\mu\nu}+i\xi D\,,\\ \delta\bar{\lambda} &= \frac{1}{2}\bar{\xi}\bar{\sigma}^\mu\sigma^\nu F_{\mu\nu}-i\bar{\xi}D\,,\\ \delta D &= \partial_\mu(\bar{\xi}\bar{\sigma}^\mu\lambda+\bar{\lambda}\bar{\sigma}^\mu\xi)\,.\end{aligned}$$

Let us write (2.1) as $\mathcal{L} \equiv (1/4\pi)(\mathrm{Im}\,\zeta)$. Furthermore, we write $\zeta = \zeta_1 + \zeta_2$, where ζ_2 contains the terms in λ and/or $F^{\mu\nu}$ and ζ_1 contains all the other terms.

We have

$$\delta\left(\partial_\mu a \partial^\mu \bar{a} - i\psi\sigma^\mu\partial_\mu\bar{\psi} + f\bar{f}\right) = \sqrt{2}\partial_\mu\left[(\xi\psi)\,\partial^\mu\bar{a} - 2\left(\bar{\psi}\bar{\sigma}^{\mu\nu}\bar{\xi}\right)\partial_\nu a + i\bar{f}\left(\bar{\xi}\bar{\sigma}^\mu\psi\right)\right]$$

so it is easy to calculate

$$\delta\zeta_1 = \sqrt{2}\partial_\mu\left[\tau\,(\xi\psi)\,\partial^\mu\bar{a} - 2\bar{\tau}\,(\xi\sigma^{\mu\nu}\psi)\,\partial_\nu\bar{a} - i\bar{\tau}f\left(\xi\sigma^\mu\bar{\psi}\right)\right].$$

To evaluate $\delta\zeta_2$, the first step is to calculate

$$\begin{aligned}\delta\left[\frac{1}{2}D^2 - \frac{1}{4}F_{\mu\nu}(F^{\mu\nu} - i\widetilde{F}^{\mu\nu}) - i\lambda\sigma^\mu\partial_\mu\bar{\lambda}\right] &= \\ = \partial^\mu\left[(\bar{\xi}\bar{\sigma}^\nu\lambda)\left(\eta_{\mu\nu}D - i(F_{\mu\nu} - i\widetilde{F}_{\mu\nu})\right)\right] &+ 2(\bar{\xi}\bar{\sigma}^\nu\lambda)(\partial^\mu\widetilde{F}_{\mu\nu}).\end{aligned}$$

When the above variation is added to the other ingredients of $\delta\zeta_2$, namely $\delta[i\sqrt{2}D(\psi\lambda) - \sqrt{2}(\lambda\sigma^{\mu\nu}\psi)F_{\mu\nu} - (\lambda\lambda)f]$ and $\delta[(\psi\psi)(\lambda\lambda)]$, a complicated expression is obtained. Following a calculation whose full details have been given elsewhere [19], we obtain

$$\delta\zeta_2 = \partial_\mu\left[\tau D\left(\bar{\xi}\bar{\sigma}^\mu\lambda\right) - i\tau\left(\bar{\xi}\bar{\sigma}_\nu\lambda\right)(F^{\mu\nu} - i\widetilde{F}^{\mu\nu}) + i\sqrt{2}\tau'\,(\psi\lambda)\left(\bar{\xi}\bar{\sigma}^\mu\lambda\right)\right] + 2\tau\left(\bar{\xi}\bar{\sigma}^\nu\lambda\right)\left(\partial^\mu\widetilde{F}_{\mu\nu}\right).$$

Finally, from $\delta\zeta_1$ and $\delta\zeta_2$ we obtain $\delta\mathcal{L}$ as given by (2.5).

After obtaining the supercurrent that generates this variation from Noether's theorem, we construct the second supercurrent by using the $\mathrm{SU}(2)_R$ symmetry of the $\mathcal{N} = 2$ vector multiplet. Under $\mathrm{SU}(2)_R$, the parameters $\xi^{(1)}$ and $\xi^{(2)}$ and the fermion fields λ and ψ transform as doublets

$$\xi^{(1)} \to \xi^{(2)}, \qquad \xi^{(2)} \to -\xi^{(1)}, \qquad \lambda \to \psi, \qquad \psi \to -\lambda,$$

while a and A^μ are singlets. These transformations induce the following transformations of the auxiliary fields:

$$f \to \bar{f}, \qquad D \to -D.$$

Alternatively, $\mathrm{SU}(2)_R$ transformations could be applied to the variations of component fields and these could be used to construct the second current directly.

References

[1] N. Seiberg and E. Witten, *Electric-magnetic duality, monopole condensation and confinement in $N = 2$ supersymmetric Yang-Mills theory*, *Nucl. Phys.* **B 426** (1994) 19 [hep-th/9407087].

[2] N. Seiberg and E. Witten, *Monopoles, duality and chiral symmetry breaking in $N = 2$ supersymmetric QCD*, *Nucl. Phys.* **B 431** (1994) 484 [hep-th/9408099].

[3] M.K. Prasad and C.M. Sommerfield, *An exact classical solution for the 't Hooft monopole and the Julia-Zee dyon*, *Phys. Rev. Lett.* **35** (1975) 760;
E.B. Bogomol'nyi, *Stability of classical solutions*, *Sov. J. Nucl. Phys.* **24** (1976) 449.

[4] E. Witten and D.I. Olive, *Supersymmetry algebras that include topological charges*, *Phys. Lett.* **B 78** (1978) 97.

[5] G. Chalmers, M. Roček and R. von Unge, *Monopoles in quantum corrected $N = 2$ super Yang-Mills theory*, hep-th/9612195.

[6] P.C. Argyres and K. Narayan, *String webs from field theory*, *J. High Energy Phys.* **03** (2001) 047 [hep-th/0101114].

[7] L. Alvarez-Gaumé and S.F. Hassan, *Introduction to s-duality in $N = 2$ supersymmetric gauge theories: a pedagogical review of the work of seiberg and Witten*, *Fortschr. Phys.* **45** (1997) 159 [hep-th/9701069].

[8] A. Iorio, *Supersymmetric noether currents and Seiberg-Witten theory*, hep-th/0006198.

[9] A. Iorio, *A note on Seiberg-Witten central charge*, *Phys. Lett.* **B 487** (2000) 171 [hep-th/9905069].

[10] S. Wolf, *Computation of the central charge for the leading order of the $N = 2$ super-Yang-Mills effective action*, *Mod. Phys. Lett.* **A 14** (1999) 2789 [hep-th/9905194].

[11] P. Srivastava, *Supersymmetry, superfields and supergravity*, Adam Hilger, 1986.

[12] M. Henneaux and C. Teitelboim, *Quantization of gauge systems*, Princeton University Press, 1992.

[13] R. Haag, J.T. Lopuszanski and M. Sohnius, *All possible generators of supersymmetries of the S matrix*, *Nucl. Phys.* **B 88** (1975) 257.

[14] J. Wess and J. Bagger, *Supersymmetry and supergravity*, 2nd edition, Princeton University Press, 1992.

[15] S. Weinberg, *The quantum theory of fields*, vol. 3: *Supersymmetry*, Cambridge University Press, 1999.

[16] D. Bailin and A. Love, *Supersymmetric gauge field theory and string theory*, IOP Publishing, 1994.

[17] E. Witten, *Dyons of charge $e\theta/2\pi$*, *Phys. Lett.* **B 86** (1979) 283.

[18] P.C. West, *Introduction to rigid supersymmetric theories*, hep-th/9805055.

[19] I.A. Popescu, *Excursions in extended supersymmetry*, Ph.D. thesis, University of Kentucky, 2001.

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